Quantum Galois theory for finite groups

Akihide Hanaki and Masahiko Miyamoto
and Daisuke Tambara

Dong and Mason [DM1] initiated a systematic research for a vertex operator algebra with a finite automorphism group, which is referred to as the "operator content of orbifold models" by physicists [DVVV]. The purpose of this paper is to extend one of their main results. We will assume that the reader is familiar with the vertex operator algebras (VOA), see [B],[FLM].

Throughout this paper, V denotes a simple vertex operator algebra, G is a finite automorphism group of V, $\mathbf{C}$ denotes the complex number field, and $\mathbf{Z}$ denotes rational integers. Let H be a subgroup of G and $Irr(G)$ denote the set of all irreducible $\mathbf{C}G$-characters. In their paper [DM1], they studied the sub VOA $V^H = \{v \in V : h(v) = v \text{ for all } h \in H\}$ of H-invariants and the subspace V^χ on which G acts according to $\chi \in Irr(G)$. Especially, they conjectured the following Galois correspondence between sub VOAs of V and subgroups of G and proved it for an Abelian or dihedral group G [DM1, Theorem 1] and later for nilpotent groups [DM2], which is an origin of their title of [DM1].

Conjecture (**Quantum Galois Theory**) Let V be a simple VOA and G a finite and faithful group of automorphisms of V. Then there is a bijection between the subgroups of G and the sub VOAs of V which contains V^G defined by the map $H \to V^H$.

Our purpose in this paper is to prove the above conjecture. Namely, we will prove:

Theorem 1 *Let V be a simple VOA and G a finite and faithful group of automorphisms of V. Then there is a bijection between the subgroups of G and the sub VOAs of V which contains V^G defined by the map H $(\leq G) \to V^H$ $(\supseteq V^G)$.*

We adopt the notation and results in [DM1] and [DLM]. Especially, the following result in [DLM] is the main tool for our study.

Theorem 2 (DLM,Corollary 2.5) *Suppose that V is a simple VOA and that G is a finite and faithful group of automorphisms of V. Then the following hold:*
(i) For $\chi \in Irr(G)$, each V^χ is a simple module for the G-graded VOA $\mathbf{C}G \otimes V^G$ of the form

$$V^\chi = M_\chi \otimes V_\chi$$

where M_χ is the simple $\mathbf{C}G$-module affording χ and where V_χ is a simple V^G-module.
(ii) The map $M_\chi \to V_\chi$ is a bijection from the set of simple $\mathbf{C}G$-modules to the set of inequivalent simple V^G-modules which are contained in V.

It was proved in [DM1, Lemma 3.2] that the map $H\ (\leq G) \to V^H\ (\supseteq V^G)$ is injective. Therefore, it is sufficient to show that for any sub VOA W containing V^G, there is a subgroup H of G such that $W = V^H$. Our first purpose is to transform the assumption of quantum Galois conjecture to the following purely group theoretic condition :

Hypotheses (A) Let G be a finite group and $\{M_\chi : \chi \in Irr(G)\}$ be the set of all nonisomorphic simple modules of G. Assume M_{1_G} is a trivial module. Let R be a subspace of $M = \oplus_{\chi \in Irr(G)} M_\chi$ containing M_{1_G}. Assume that for any G-homomorphism $\pi : M \otimes M \to M$, $\pi(R \otimes R) \subseteq R$.

Let's show how to transform the assumption of quantum Galois conjecture to Hypotheses (A). We first explain the way to give a relation between V and M. Let introduce a relation $u \sim v$ for two nonzero elements $u, v \in V$ if there is an element $w \in V^G$ such that $v = w_n u$ or $u = w_n v$ for some $n \in Z$. Extend this relation into an equivalent relation $\equiv$ as follows:
$u \equiv v$ if there are $u^1, ..., u^m \in V$ such that $u \sim u^1 \sim ... \sim u^m \sim v$.
This equivalent relation implies that for any G-homomorphism $\phi : V \to M_\chi$, the image of each equivalent class is uniquely determined up to scalar times. Also since V_χ is spanned by $\{v_n s_\chi : v \in V^G, n \in \mathbf{Z}\}$ for some $s_\chi \in V_\chi$ by [DM1 Proposition 4.1], $\phi(V_\chi) = \mathbf{C}\phi(s_\chi)$ is a subspace of dimension at most one.

Let's start the transformation. We recall

$$V = \oplus_\chi (M_\chi \otimes V_\chi)$$

from Theorem 2 ([DLM, Corollary 2.5]). We may view M_χ as a subspace of V. Set $M = \oplus M_\chi$. Let W be a sub VOA containing V^G. Let U_χ be the V^G-subspace of W whose composition factors are all isomorphic to V_χ. Then $U_\chi = W \cap (M_\chi \otimes V_\chi)$ and so $W = \oplus_\chi U_\chi$. Set

$$R_\chi = \{m \in M_\chi | m \otimes V_\chi \subseteq W\}$$

and $R = \oplus R_\chi$. In particular, we have

$$W = \oplus_\chi (R_\chi \otimes V_\chi).$$

Replacing M_χ by $L(k_1)...L(k_r)M_\chi \cong M_\chi$ if necessary, we may think that $R = \oplus R_\chi$ and $M = \oplus M_\chi$ are subspaces of a homogeneous part V_p of V for some p. Let $\{v^1, ..., v^s\}$ be a basis of R and $\{v^1, ..., v^s, ..., v^n\}$ be a basis of M. The proof of Lemma 3.1 in [DM1] shows that $\{Y(v^i, z)v^j \,:\, i, j = 1, ..., n\}$ is a linearly independent set. Define $\pi_m : M \times M \to V_m$ by $\pi_m(v^i \times v^j) = (v^i)_m v^j$. Since $\{Y(v^i, z)v^j\}$ is a linearly independent set, $\cap_{m \in \mathbf{Z}} \mathrm{Ker}(\pi_m) = 0$. Since $M \times M$ has only a finite dimension, there is a finite set $\{a, a+1, ..., b\}$ of integers such that $\cap_{m=a}^b \mathrm{Ker}(\pi_m) = 0$. Since the grade of $(v^i)_m(v^j)$ depends only on m and $(v^i)_m(v^j)$ and $(v^h)_{m'}(v^k)$ belong to different homogeneous spaces for $m \neq m'$, the map $\pi : M \times M \to V$ given by $\pi(v^i \times v^j) = \sum_{m=a}^b \{(v^i)_m(v^j)\}$ is injective. Set $E = \mathrm{Im}(\pi) =< \sum_{m=a}^b (v^i)_m(v^j) : i, j = 1, ..., n >$. Clearly, E is a G-invariant subspace of V. Decompose V into a direct sum $V = E \oplus E'$ of E and some G-submodule E' of V. Define $\mu : V = E \oplus E' \to M \times M$ by $\mu(e + e') = \pi^{-1}(e)$ for $e \in E, e' \in E'$. This is a G-epimorphism. Since W is a sub VOA, the any products $u_n v$ of two elements u, v of W are in W. Hence, $\pi(R \times R) \subseteq W$ and so the image $\mu(W)$ contains $R \times R$. Therefore, for any G-homomorphism $\phi : M \otimes M \to M$, we have $\phi(R \times R) \subseteq \phi(\mu(W))$. On the other hand, for any G-homomorphism $\psi : V \to M$, we have $\psi(W) \subseteq R$ since $W = \oplus_\chi (R_\chi \otimes V_\chi)$. Hence, we have

$$\phi(R \times R) \subseteq \phi(\mu(W)) = (\phi\mu)(W) \subseteq R$$

for any G-homomorphism $\phi : M \otimes M \to M$. Namely, R satisfies the Hypotheses (A).

We will next prove the following group theoretic problem:

Theorem 3 *Let G be a finite group and $\{M_\chi : \chi \in Irr(G)\}$ be the set of all simple modules of G. Assume $M_1 = \mathbf{C}$ is a trivial module. Let R be a subspace of $M = \oplus_{\chi \in Irr(G)} M_\chi$ containing M_1. Assume that R satisfies the following condition: for any G-homomorphism $\pi : M \otimes M \to M$, $\pi(R \otimes R) \subseteq R$. Then there is a subgroup G_1 of G such that $R = M^{G_1}$.*

Proof. Consider the group algebra $\mathbf{C}G$, and write $\mathbf{C}G = \oplus_{\chi \in Irr(G)} {M_\chi}^{\chi(1)}$ as a left G-module. We define a subspace S of $\mathbf{C}G$ by

$$S = \bigoplus_{\chi \in Irr(G)} (R \cap M_\chi)^{\chi(1)}.$$

Then S is a right ideal of $\mathbf{C}G$. Note that $R = \oplus_\chi (R \cap M_\chi)$, since $M_1 = \mathbf{C} \subseteq R$ and $\pi_\chi(\mathbf{C} \otimes R) \subseteq R$ for a projection $\pi_\chi : \mathbf{C} \otimes M \to M_\chi \subseteq M$. Thus S satisfies that $\pi'(S \otimes S) \subseteq S$ for any G-homomorphism $\pi' : \mathbf{C}G \otimes \mathbf{C}G \to \mathbf{C}G$.

We define a new product $\circ$ in $\mathbf{C}G$ by

$$\left(\sum_{g \in G} a_g g\right) \circ \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} a_g b_g g,$$

where a_g, $b_g \in \mathbf{C}$. Then $(\mathbf{C}G, \circ)$ is a semisimple commutative algebra with the identity $\sum_{g \in G} g$. We write the identity by 1°.

Define $\pi' : \mathbf{C}G \otimes \mathbf{C}G \to \mathbf{C}G$ by

$$\pi'\left(\left(\sum_{g \in G} a_g g\right) \otimes \left(\sum_{g \in G} b_g g\right)\right) = \sum_{g \in G} a_g b_g g.$$

Then π' is a G-homomorphism, and so $\pi'(S \otimes S) \subseteq S$. This means that S is a subalgebra of $(\mathbf{C}G, \circ)$.

Since R contains the trivial module, 1° belongs to S. Consider the primitive idempotent decomposition of 1° in S. Then there exists a partition $G = \cup_i G_i$ such that $\sum_{g \in G_i} g$ is a primitive idempotent in S for any i. Put $e_i = \sum_{g \in G_i} g$, then $S = \oplus_i \mathbf{C} e_i$ since S is semisimple and commutative. Assume $1_G \in G_1$. For $h \in G_1$, $e_1 h$ is in S since S is a right ideal in $\mathbf{C}G$. By the form of S, $e_1 h$ is a sum of some e_i's. But $h \in G_1 h$, so $e_1 h = e_1$. This means G_1 is a subgroup of G. Similarly G_i is a left coset of G_1 in G for any i.

Now $S = \mathbf{C}G^{G_1}$, and so $R = V^{G_1}$. The proof is completed. $\square$

Let's go back to the proof of Theorem 1 (the quantum Galois theory). By the above theorem 3, there is a subgroup H of G such that $R_\chi = M_\chi^H$ and so

$$U_\chi = M_\chi^H \otimes V_\chi = (M_\chi \otimes V_\chi)^H.$$

Hence, $W = V^H$. This completes the proof of Theorem 1.

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Akihide Hanaki
Faculty of Engineering, Yamanashi University, Kofu 400, Japan

Masahiko Miyamoto
Institute of Mathematics, University of Tsukuba, Tsukuba 305, Japan

Daisuke Tambara
Department of Mathematics, Hirosaki University, Hirosaki 036, Japan